

Proximinal and Čebyšev Sets in Normed Linear Spaces

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Abstract. In this paper, we study a part of approximation theory that presents the conditions under which a closed set in a normed linear space is proximinal or Čebyšev.

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1. BASIC DEFINITIONS AND PRELIMINARIES

In this section, we collect some definitions which will help us to describe our results in detail. As the first step, let us fix our notation. Through this paper, K denotes a non-empty subset of real normed linear space $(X, \|\cdot\|)$ with the topological dual space X^* , $S(X) = \{x \in X; \|x\| = 1\}$, $B[x; r] = \{y \in X; \|y - x\| \leq r\}$ and $B(X) = B[0; 1]$.

For an element $x \in X$, we define the distance function $d_K : X \rightarrow \mathbb{R}$ by $d_K(x) = \inf\{\|y - x\|; y \in K\}$. It is easy to see that the value of $d_K(x)$ is zero if and only if x belongs to \overline{K} , the closure of K . The subset K is called proximinal (resp. Čebyšev), if for each $x \in X \setminus K$, the set of best approximations to x from K

$$P_K(x) = \{y \in K; \|y - x\| = d_K(x)\},$$

is nonempty (resp. a singleton). This concept was introduced by S. B. Stechkin and named after the founder of best approximation theory, Čebyšev.

It is interesting to know the sufficient conditions for a subset K of a given normed linear space to be a proximal or a Čebyšev set, and this is what we want to consider in this paper.

It is not difficult to show that every proximal subset K of X is also closed. Now, we state and prove a sufficient condition for proximality:

If K is a closed subset of a finite-dimensional space X , then K is proximal. To see this, suppose that $x_0 \in X \setminus K$ and $r_0 = d_K(x_0)$. If $r > r_0$, then there exists $y \in K$ such that $\|x_0 - y\| < r$. Therefore $y \in B[x_0; r] \cap K$. It follows that $B[x_0; r] \cap K \neq \emptyset$. If $B_n = B[x_0; r_0 + \frac{1}{n}] \cap K$, then it is clear that B_n is a non-empty compact subset of X and $B_{n+1} \subseteq B_n$ for all $n \geq 1$. Hence, there exists $y_0 \in X$ such that $y_0 \in \bigcap_{n=1}^{\infty} B_n$. Now, we have $\|y_0 - x_0\| \leq r_0 + \frac{1}{n}$ for all $n \geq 1$. Since $r_0 = d_K(x_0)$ we have $\|y_0 - x_0\| = r_0 = d_K(x_0)$. Thus y_0 is a best approximation for x_0 and therefore K is a proximal set.

In general, since the functional $e_x : K \rightarrow \mathbb{R}$ with $e_x(y) = \|y - x\|$ is continuous, each compact subset of X is proximal.

It is easy to see that in a reflexive space, every weakly closed set is proximal.

Question. Is there a closed nonempty subset K of a reflexive Banach space X with the property that no point outside K admits a best approximation in K ? Is this possible in an equivalent renorm of a Hilbert space? The Lau-Konjagin theorem (see [2]) states that in a reflexive Banach space X , for every closed set K there is a dense set in $X \setminus K$ which admits best approximations if and only if the norm has the Kadec-Klee property. (i. e. for each sequence $(x_n)_{n=1}^{\infty}$ in X which converges weakly to x with $\lim_{n \rightarrow \infty} \|x_n\| = \|x\|$, we have $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$).

Every closed convex set in a reflexive space is proximal [2]. However, this theorem is not true in the absence of reflexivity. In fact, this condition is a sufficient one. See the following example:

Let $X = l^1$. It is known that l^1 is a non-reflexive Banach space with dual space l^∞ . For any positive integer n , let $e_n \in l^1$ be such that its n th entry is $\frac{(n+1)}{n}$ and all other entries are 0. Let $K = \overline{\text{co}}\{e_1, e_2, \dots, e_n, \dots\}$. Then K is a closed convex subset of l^1 and is not proximal.

Another important notion in this paper is metric projection. The metric projection mapping has been used in many areas of mathematics such as the theory of optimization and approximation, and fixed point theory. It is a set-valued mapping $P_K : X \rightarrow K$ which associates to each x in X the set of all its best approximations, namely $P_K(x)$. The sequence $(y_n)_{n=1}^\infty \subseteq K$ is called minimizing for $x \in X \setminus K$ if, $\lim_{n \rightarrow \infty} \|x - y_n\| = d_K(x)$ and we say that the metric projection P_K is continuous at $x \in X \setminus K$ provided that $\lim_{n \rightarrow \infty} y_n = y_0$ if, $y_n \in P_K(x_n)$ for each $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} x_n = x_0$. It is clear that P_K is continuous at x if, every minimizing sequence for $x \in X \setminus K$ converges [10]. The continuity properties of P_K is a natural object of study in understanding the nature of some problems in approximation theory. In the linear cases many results show the connection of the continuity properties and the geometry of the Banach space (see [12]). We use this property to prove our main result.

In order to give sufficient conditions for a set being proximinal, N. V. Efimov and S. B. Stechkin introduced the concept of approximatively compact sets. The set K is said to be approximatively compact if, for any $x \in X$, each minimizing sequence $(y_n)_{n=1}^\infty \subseteq K$ for x has a subsequence converging to an element of K . It is proved in [12] that every approximatively compact set is proximinal.

We say that K is boundedly compact, provided that $K \cap B[0; r]$ is compact in X for every $r \geq 0$. Every boundedly compact set is approximatively compact although the converse is false. Thus, every boundedly compact set is proximinal, too.

Let $f : X \rightarrow \mathbb{R}$ be a function and $x, y \in X$. Then f is said to be Gateaux differentiable at x if, there exists $A \in X^*$ such that $A(y) = \lim_{t \rightarrow 0} \frac{f(x + ty) - f(x)}{t}$. In this case A is called the Gateaux derivative of f and is denoted by $f'(x)$. Also, $A(y)$ is denoted by $\langle f'(x), y \rangle$, usually. If the above limit exists uniformly for each $y \in S(X)$, then f is said to be Fréchet differentiable at x with Fréchet derivative A . Similarly, the norm function $\|\cdot\|$ is Gateaux (Fréchet) differentiable at $0 \neq x \in X$ if, the function $f(x) = \|x\|$ is Gateaux (Fréchet) differentiable.

It is well known that if, $f : X \rightarrow \mathbb{R}$ is Fréchet differentiable at $x \in X$ then for given $\varepsilon > 0$ there exists $\delta_{(x, \varepsilon)} > 0$ such that $\|f(x + y) - f(x) - \langle f'(x), y \rangle\| \leq \varepsilon \|y\|$, for each $y \in X$ with $\|y\| < \delta$.

2. MAIN RESULTS

We start our work with the following lemma:

Lemma 1. Suppose K is closed and the distance function d_K is Gateaux differentiable at $x \in X \setminus K$. Then for every $y \in P_K(x)$ we have $\langle d'_K(x), \frac{x-y}{\|x-y\|} \rangle = 1$.

Proof. At first, from Gateaux differentiability of d_K , the limit

$$\liminf_{t \rightarrow 0^+} \frac{d_K(x + tz) - d_K(x)}{t},$$

exists for every $z \in X$. But for each $t > 0$

$$d_K(x + t(x - y)) - d_K(x) \leq td_K(x).$$

Hence, in particular, for $z = x - y$

$$\liminf_{t \rightarrow 0^+} \frac{d_K(x + tz) - d_K(x)}{t} = d_K(x).$$

Now if $t' = \frac{t}{d_K(x)}$ (notice that $d_K(x) > 0$) then

$$\liminf_{t' \rightarrow 0^+} \frac{d_K(x + t'(x - y)) - d_K(x)}{t'} = d_K(x),$$

and consequently

$$\liminf_{t \rightarrow 0^+} \frac{d_K(x + t \frac{x-y}{\|x-y\|}) - d_K(x)}{t} = 1.$$

On the other hand, since distance functions are Lipschitz (with constant 1) we have

$$\limsup_{t \rightarrow 0^+} \frac{d_K(x + t \frac{x-y}{\|x-y\|}) - d_K(x)}{t} \leq 1,$$

as required.

We say that a non-zero element $x^* \in X^*$ strongly exposes $B(X)$ at $x \in S(X)$, provided a sequence $(z_n)_{n=1}^\infty$ in $B(X)$ converges to x whenever $(\langle x^*, z_n \rangle)_{n=1}^\infty$ converges to $\langle x^*, x \rangle$.

The following theorem is the same as theorem 2.6 in [10], but with some manipulation, and plays a key role in our work:

Theorem 2. Suppose K is closed in X and d_K is Fréchet differentiable at $x \in X \setminus K$. Moreover $y \in P_K(x)$ and $d'_K(x)$ strongly exposes $B(X)$ at $\|x - y\|^{-1}(x - y)$. Then every minimizing sequence $(y_n)_{n=1}^\infty$ in K for x converges to y .

Proof. We can choose a sequence $(a_n)_{n=1}^{\infty}$ of positive numbers such that $\lim_{n \rightarrow \infty} a_n = 0$ and

$$a_n^2 > \|x - y_n\| - d_K(x) \quad (n \in \mathbb{N}).$$

Hence, if $0 < t < 1$ then for each $n \in \mathbb{N}$

$$\begin{aligned} d_K(x + t(y_n - x)) &\leq \|x + t(y_n - x) - y_n\| \\ &= (1 - t)\|x - y_n\| \\ &< (1 - t)(a_n^2 + d_K(x)). \end{aligned}$$

Therefore

$$d_K(x) - d_K(x + t(y_n - x)) \geq t d_K(x) - 2a_n^2.$$

Fix $\varepsilon > 0$. By Fréchet differentiability of d_K , there is $\delta > 0$ such that if $\|y\| < \delta$ then

$$|d_K(x + y) - d_K(x) - \langle d'_K(x), y \rangle| \leq \varepsilon \|y\| \quad (*).$$

Let $t_n = \frac{a_n}{\|x - y_n\|}$ and $a_n < \delta$ for large n . Replacing y by $t_n(y_n - x)$ in $(*)$ we get

$$\begin{aligned} \varepsilon t_n \|x - y_n\| - \langle d'_K(x), t_n(y_n - x) \rangle &\geq d_K(x) - d_K(x + t_n(y_n - x)) \\ &\geq t_n d_K(x) - 2a_n^2, \end{aligned}$$

whence

$$\langle d'_K(x), t_n(x - y_n) \rangle \geq -\varepsilon a_n - 2a_n^2 + t_n d_K(x),$$

therefore

$$\langle d'_K(x), \|x - y_n\|^{-1}(x - y_n) \rangle \geq -\varepsilon - 2a_n + \frac{d_K(x)}{\|x - y_n\|}.$$

Since $\varepsilon > 0$, $\lim_{n \rightarrow \infty} a_n = 0$, $\lim_{n \rightarrow \infty} \|x - y_n\| = d_K(x)$, we will have

$$1 \geq \liminf_{n \rightarrow \infty} \langle d'_K(x), \|x - y_n\|^{-1}(x - y_n) \rangle \geq \liminf_{n \rightarrow \infty} \frac{d_K(x)}{\|x - y_n\|} = 1,$$

therefore by the lemma 1

$$\lim_{n \rightarrow \infty} \langle d'_K(x), \|x - y_n\|^{-1}(x - y_n) \rangle = 1 = \langle d'_K(x), \|x - y\|^{-1}(x - y) \rangle.$$

Since $d'_K(x)$ strongly exposes $B(X)$ at $\|x - y\|^{-1}(x - y)$, we deduce that

$$\lim_{n \rightarrow \infty} \|x - y_n\|^{-1}(x - y_n) = \|x - y\|^{-1}(x - y),$$

which yields $\lim_{n \rightarrow \infty} y_n = y$.

It is interesting to know that if K is closed in X , $x \in X \setminus K$ and $(y_n)_{n=1}^{\infty}$ is a

minimizing sequence in K for x with the weak limit $y \in K$, then y is a best approximation for x in K . This is because the norm is a lower-semi-continuous function with respect to weak topology and we have

$$d_K(x) \leq \|x - y\| \leq \liminf_{n \rightarrow \infty} \|x - y_n\| \leq \lim_{n \rightarrow \infty} \|x - y_n\| = d_K(x).$$

Theorem 3. [6] The dual norm of X^* is Fréchet differentiable at $x^* \in X^*$ if and only if x^* strongly exposes $B(X)$.

Corollary 4. Let K is closed in X , the distance function d_K is Fréchet differentiable at $x \in X \setminus K$ and the dual norm of X^* is Fréchet differentiable. Then each minimizing sequence in K for x is convergent.

Proof. Combine Theorem 2 with Theorem 3.

Corollary 5. Let K be closed in X , $x \in X \setminus K$ and the distance function d_K is Fréchet differentiable at x . Also, assume that the dual norm of X^* is Fréchet differentiable. Then the metric projection P_K is continuous at x .

We say that the space X is strictly convex (rotund) if, $x = y$ whenever $\|x\| = \|y\| = \|\frac{x+y}{2}\| = 1$ and X is called uniformly convex if, for sequences $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \subseteq X$ with

$$\lim_{n \rightarrow \infty} 2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 = 0,$$

we have

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0.$$

Obviously, uniformly convex Banach spaces are strictly convex and also, they are reflexive (Milman-Pettis).

Remark 6. It is a well known theorem that the dual norm of X^* is Fréchet differentiable if and only if X is uniformly convex. Therefore, we have the following corollary.

Corollary 7. Suppose that K is closed in a uniformly convex space X , $x \in X \setminus K$ and d_K is Fréchet differentiable at x . Then the metric projection P_K is continuous at x .

We can say also about weakly closed sets that, each weakly closed set in a uniformly convex Banach space has continuous metric projection [8].

It is proved that closed convex sets in strictly convex reflexive Banach spaces (and consequently in uniformly convex Banach spaces) are Čebyšev (see [6]). Can we prove that in some Banach spaces, a nonempty subset is a Čebyšev

set if and only if it is closed and convex? This is an open problem, even in the special case of infinite-dimensional Hilbert space (see [4]). In 1934, L. N. H. Bunt proved that each Čebyšev set in a finite-dimensional Hilbert space must be convex. From this result, we see that in a finite-dimensional Hilbert space, a nonempty subset is a Čebyšev set if and only if it is closed and convex. In [11], G. G. Johnson gave an example: there exists an incomplete inner product space which possesses a non-convex Čebyšev set (M. Jiang completed the proof in 1993). Is there an infinite-dimensional Hilbert space possessing a non-convex Čebyšev set? As addressed above, it is unknown. Now, in the last part of the paper, we present a condition under which a closed subset is Čebyšev.

It can be seen in [8] that if the dual norm of X^* is Fréchet differentiable, then the closed sets in X with continuous metric projection are Čebyšev.

Finally, the following is immediate from corollary 7.

Corollary 8. Let K be closed in a uniformly convex Banach space X , $x \in X \setminus K$ and d_K is Fréchet differentiable at x . Then K is Čebyšev in X .

Corollary 8 is also valid in infinite-dimensional Hilbert spaces.

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